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An analysis of a perfectly conducting capacitive circuit

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Abstract. The plates of a charged capacitor are connected by a loop of perfectly conducting wire. An integral–differential equation is derived for the charge on the capacitor and an existence and uniqueness result for the solution of the corresponding initial value problem is demonstrated. A short example illustrates the behaviour of the current in the circuit.

Introduction

The plates of a charged capacitor are connected by a loop of perfectly conducting wire. The behaviour of the current in the resulting circuit is difficult to determine by elementary means. It is natural to attempt to analyse the problem as an LCR_{rad} circuit where L is geometric self-inductance, C is capacitance and R_{rad} is radiation resistance. However, radiation resistance is usually computed for a harmonically driven circuit and its value depends on the frequency. The circuit described above is not driven; even if one assumes an alternating current, the frequency is unknown. An analysis along these lines would require non-trivial assumptions on the behaviour of the current and would not be rigorous. (For more on the application of elementary methods, one can consult the literature on the closely related two-capacitor problem, e.g. Cuvaj 1968 and Powell 1979.) This paper analyses the circuit starting from Maxwell's equations or, more accurately, the equivalent reformulation of those equations in terms of vector and scalar potentials. Assuming uniformity of current, an integral–differential equation is derived for the charge on the capacitor. The existence and uniqueness of the solution of the corresponding initial value problem is demonstrated, provided the diameter of the circuit is not too large. The solution current is difficult to compute explicitly but it can be approximated in light of this result.

The body of the paper is in three sections. In the first section assumptions are listed and the equation for the charge on the capacitor is derived. In the second section the existence and uniqueness result is demonstrated. The third section outlines a computation illustrating the behaviour of such a circuit.

1. Assumptions and derivation of equation for current

The MKSA system of units is employed. Time derivatives are denoted by dots over variables. Notation is standard (Jackson 1975).

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The parallel-plate capacitor has capacitance, C, and is without dielectric. The plates are, say, very thin identical disks and the dimensions of the capacitor are small in comparison to the radius, r_0, of the circular loop of connecting wire. In particular, the gap between the capacitor plates is very small. The wire is attached at the centres of the plates. The cross-sectional radius of the wire is r_1, $r_1 \ll r_0$. The current, I, is assumed to be uniform within the perfectly conducting circuit and the charge density, ρ, is assumed to be uniform on the plates.

Expressing the electric and magnetic induction fields $\boldsymbol{E}$ and $\boldsymbol{B}$ in terms of vector and scalar potentials $\boldsymbol{A}$ and Φ belonging to the Lorentz gauge yields

(i) $\boldsymbol{B} = \nabla \times \boldsymbol{A}$

(ii) $\boldsymbol{E} = -\dot{\boldsymbol{A}} - \nabla \Phi$

(iii) $$\boldsymbol{A}(\boldsymbol{x}, t) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}\left(\boldsymbol{x}', t - \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right)}{|\boldsymbol{x}-\boldsymbol{x}'|} \, \mathrm{d}^3 x'$$

(iv) $$\Phi(\boldsymbol{x}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho\left(\boldsymbol{x}', t - \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right)}{|\boldsymbol{x}-\boldsymbol{x}'|} \, \mathrm{d}^3 x'$$

Here $\boldsymbol{J}$ is the current density and μ_0, ε_0 and c are the permeability, permittivity and speed of light constants. With regards to initial conditions, it is assumed that $\boldsymbol{J}(\boldsymbol{x}, t) = \boldsymbol{0}$ for $t \leqslant 0$ and $\rho(\boldsymbol{x}, t) = \rho_0(\boldsymbol{x})$ for $t \leqslant 0$. Here $\rho_0(\boldsymbol{x})$ is the charge density on the capacitor plates corresponding to the initial charge, Q_0. Note that since the capacitor plates are thin, the charge is distributed over an essentially two-dimensional region in three-space. Thus, ρ is interpreted in a distributional sense. However, $\boldsymbol{J}$ is not. It is important to view the connecting wire as three-dimensional and not as a one-dimensional filament. With a filament the magnetic field (and hence the flux of induction) would be infinite and the analysis would break down.

To pass from the field equations to an equation for the current, the wire is regarded as a 'bundle' of circles (with very small gap) of infinitesimal cross-sectional area $\mathrm{d}s$. Computing the line integral of each side of (ii) over such a circle yields

$$0 = \oint \dot{\boldsymbol{A}}(\boldsymbol{x}, t) \cdot \mathrm{d}\boldsymbol{l} + (\Phi(\boldsymbol{x}_1, t) - \Phi(\boldsymbol{x}_2, t)). \tag{1}$$

Here $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ are the initial and terminal points of the circle at the gap. Let r denote $|\boldsymbol{x}-\boldsymbol{x}'|$, the distance from the field point to the source point. Since the capacitor is small with respect to the circuit, at the field points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ one can ignore the retardation rc^{-1} in the time variable in the integral in (iv), i.e. at a field point within the capacitor the retardation effects in (iv) are negligible compared to the retardation effects in (iii) at any field point within the circuit. Thus, the difference $\Phi(\boldsymbol{x}_1, t) - \Phi(\boldsymbol{x}_2, t)$ is approximately $C^{-1}Q(t)$ where $Q(t)$ is the charge on the capacitor. Using this approximation, (iii) and the uniform current assumption, and then integrating with respect to $\mathrm{d}s$ over the cross-section, S, of the wire, one obtains

$$0 = \frac{\mu_0}{4\pi(\pi r_1^2)^2} \int_S \int_S \oint \oint \frac{\dot{I}(t - rc^{-1})}{r} \, \mathrm{d}\boldsymbol{l}' \cdot \mathrm{d}\boldsymbol{l} \, \mathrm{d}s' \, \mathrm{d}s + C^{-1}Q(t). \tag{2}$$

Equivalently, using $I = \dot{Q}$,

$$0 = \frac{\mu_0}{4\pi(\pi r_1^2)^2} \int_S \int_S \oint \oint \frac{\ddot{Q}(t - rc^{-1})}{r} \, \mathrm{d}\boldsymbol{l}' \cdot \mathrm{d}\boldsymbol{l} \, \mathrm{d}s' \, \mathrm{d}s + C^{-1} Q(t). \tag{3}$$

Observe that there are two initial conditions associated with (3), namely

$$Q(0) = Q_0 \qquad \text{and} \qquad \dot{Q}(0) = 0. \tag{4}$$

2. Existence and uniqueness of solutions

Let $L(z)$ be the analytic function defined by

$$L(z) := \frac{\mu_0}{4\pi(\pi r_1^2)^2} \int_S \int_S \oint \oint \frac{\mathrm{e}^{-zr/c}}{r} \, \mathrm{d}\boldsymbol{l}' \cdot \mathrm{d}\boldsymbol{l} \, \mathrm{d}s' \, \mathrm{d}s. \tag{5}$$

A function of the form $Q(t) = \mathrm{e}^{zt}$, $z \in \mathbb{C}$, is a solution of (3) if and only if z satisfies

$$L(z) z^2 + C^{-1} = 0. \tag{6}$$

It is clear that solutions of (6) cannot be real, and since $\overline{L(z)} = L(\bar{z})$, it follows that solutions occur in distinct complex conjugate pairs.

Let L_0 denote the quantity

$$L_0 := \frac{\mu_0}{4\pi(\pi r_1^2)^2} \int_S \int_S \oint \oint \frac{1}{r} \, \mathrm{d}\boldsymbol{l}' \cdot \mathrm{d}\boldsymbol{l} \, \mathrm{d}s' \, \mathrm{d}s \tag{7}$$

and set

$$\begin{aligned} f(z) &:= L_0 z^2 + C^{-1} \\ g(z) &:= L(z) z^2 + C^{-1}. \end{aligned} \tag{8}$$

Let ω_0 *denote the quantity* $(L_0 C)^{-1/2}$ *and observe that* f *has precisely two zeros,* $\pm i\omega_0$. Rouché's theorem (Ahlfors 1979) will imply that g has precisely two zeros (complex conjugates as observed earlier) enclosed by the circle $|z| = M\omega_0$ if

$$M > 1 \qquad \text{and} \tag{9}$$

$$|g(z) - f(z)| < |f(z)| \qquad \text{for all } |z| = M\omega_0. \tag{10}$$

Inequality (10) is satisfied if the following holds for all $|z| = M\omega_0$:

$$\left| \frac{\mu_0}{4\pi(\pi r_1^2)^2} \int_S \int_S \oint \oint \frac{\mathrm{e}^{-zr/c} - 1}{r} \, \mathrm{d}\boldsymbol{l}' \cdot \mathrm{d}\boldsymbol{l} \, \mathrm{d}s' \, \mathrm{d}s \right| < L_0 - C^{-1}(M\omega_0)^{-2}. \tag{11}$$

For $N > M$, the complex version of Taylor's theorem (Ahlfors 1979) implies

$$\mathrm{e}^{-zr/c} - 1 = \frac{z}{2\pi \mathrm{i}} \oint_{|\xi| = N\omega_0} \frac{\mathrm{e}^{-\xi r/c}}{\xi(\xi - z)} \, \mathrm{d}\xi \tag{12}$$

which for $|z| = M\omega_0$ implies

$$|e^{-zr/c} - 1| \leqslant \frac{M}{N-M} e^{N\omega_0 r/c}. \tag{13}$$

If ε denotes the quantity $M\omega_0 r_0/c$, it follows from (11) and (13) that (9) and (10) are satisfied if $M > 1$ and

$$\varepsilon < \frac{M}{2N} \ln\left(\frac{(N-M)(M^2-1)}{M^3}\right) \tag{14}$$

or choosing, for example, $N = 3M = 6$,

$$\varepsilon \leqslant \tfrac{1}{15} < \tfrac{1}{6} \ln \tfrac{3}{2}. \tag{15}$$

Thus, there exist two distinct solutions, $Q_1(t) = e^{zt}$, $Q_2(t) = e^{\bar{z}t}$, $|z| < M\omega_0$, of (3) if

$$r_0 \leqslant \frac{c}{30\omega_0}. \tag{16}$$

A bound on r_0 such as in (16) is not unreasonable considering the uniform current assumption. Observe that ε serves as an upper bound for the ratio of the circumference of the circuit and the wavelength of the emitted radiation. Thus (15) and (16) are in the same spirit as the Hertzian dipole approximation (Marion and Hornyak 1982). Note that the right side of (16) depends on r_0 (and r_1) via $\sqrt{L_0}$.

A solution of the initial value problem (3), (4) is obtained by taking linear combination of $Q_1(t)$ and $Q_2(t)$. If $z = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$, $\beta > 0$, this solution is

$$Q(t) = Q_0 e^{\alpha t}(\cos \beta t - \alpha\beta^{-1} \sin \beta t) \tag{17}$$

with corresponding current

$$I(t) = -\frac{Q_0(\alpha^2 + \beta^2)}{\beta} e^{\alpha t} \sin \beta t. \tag{18}$$

It is easy to check that (15), (5) and (6) imply $\alpha < 0$. Thus, the current decays as it oscillates.

Laplace transform techniques can be used to demonstrate the uniqueness of the solution of the initial value problem. Let $q(s)$ denote the transform of $Q(t)$. Applying the transform to (3) and using the initial conditions (4) (bearing in mind $Q(t) = Q_0$ for $t \leqslant 0$) yields

$$q(s) = \frac{Q_0 s L(s)}{s^2 L(s) + C^{-1}}. \tag{19}$$

The right side of (19) is independent of $q(s)$; it follows from Lerch's theorem (Widder 1941) that the solution (17) is the unique solution with, say, a piecewise continuous second derivative of exponential order.

3. An example

For values of C, r_0 and r_1 satisfying (15), it is difficult to find an exact expression for $Q(t)$ in the form of (17). There is an explicit integral expression for $Q(t)$ in terms of (19) and the complex inversion formula:

$$Q(t) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{-iT}^{iT} q(s) e^{st} \, ds. \tag{20}$$

However, to solve for $Q(t)$ via (20) is not practical. To evaluate the right-hand side using the calculus of residues would require knowledge of the zeros of the denominator of $q(s)$, i.e. the solutions of (6). Thus, it is more efficient to find the solutions $\alpha \pm i\beta$ of (6) directly and then use them in (17) or (18). Solving (6) is not simple, but the existence and uniqueness result of the previous section permits one to approximate the solutions.

Using Euler's formula and terms from the series expansions of the exponential, cosine and sine functions about the origin, one finds that the integrand in (5) at $z=\alpha+i\beta$ is

$$\approx \frac{1}{r}\left(1-\frac{\alpha r}{c}\right)\left[\left(1-\frac{\beta^2 r^2}{2c^2}\right)-i\left(\frac{\beta r}{c}-\frac{\beta^3 r^3}{6c^3}\right)\right]$$

$$\approx \frac{1}{r}\left(1-\frac{\alpha r}{c}-\frac{\beta^2 r^2}{2c^2}+\frac{\alpha\beta^2 r^3}{2c^3}\right)-i\frac{1}{r}\left(\frac{\beta r}{c}-\frac{\alpha\beta r^2}{c^2}-\frac{\beta^3 r^3}{6c^3}\right). \tag{21}$$

Since the gap between the plates is very small, the line integrals in (5) can be approximated by integrals over a full circle. Line integrals of constants over closed contours are zero, hence

$$\operatorname{Re} L(\alpha+i\beta) \approx \frac{\mu_0}{4\pi(\pi r_1^2)^2}\int_S\int_S \oint\oint \frac{1}{r}\left(1-\frac{\beta^2 r^2}{2c^2}+\frac{\alpha\beta^2 r^3}{2c^3}\right) d\mathbf{l}'\cdot d\mathbf{l}\, ds'\, ds \tag{22}$$

$$\operatorname{Im} L(\alpha+i\beta) \approx \frac{\mu_0}{4\pi(\pi r_1^2)^2}\int_S\int_S \oint\oint \frac{1}{r}\left(\frac{\alpha\beta r^2}{c^2}+\frac{\beta^3 r^3}{6c^3}\right) d\mathbf{l}'\cdot d\mathbf{l}\, ds'\, ds. \tag{23}$$

Note that $|\alpha|, |\beta| < M\omega_0$ so the terms within the parentheses containing α or β are of the order of ε^2 or ε^3. Thus,

$$|\operatorname{Im} L(\alpha+i\beta)| \ll |\operatorname{Re} L(\alpha+i\beta)| \tag{24}$$

$$\operatorname{Re} L(\alpha+i\beta) \approx \frac{\mu_0}{4\pi(\pi r_1^2)^2}\int_S\int_S \oint\oint \frac{1}{r}\, d\mathbf{l}'\cdot d\mathbf{l}\, ds'\, ds \tag{25}$$

$$\approx \mu_0 r_0\left[\ln\left(\frac{8r_0}{r_1}\right)-\frac{7}{4}\right] \qquad r_1 \ll r_0. \tag{26}$$

The integral in (25) is the geometric self-inductance of a wire loop. For this and the approximation (26), see Becker and Sauter (1982). Note that the quantities in (25) and (26) also approximate L_0.

Now, for $r_1 \ll r_0$

$$\frac{1}{(\pi r_1^2)^2}\int_S\int_S \oint\oint r\, d\mathbf{l}'\cdot d\mathbf{l}\, ds'\, ds \approx -\frac{16\pi}{3} r_0^3 \tag{27}$$

$$\frac{1}{(\pi r_1^2)^2}\int_S\int_S \oint\oint r^2\, d\mathbf{l}'\cdot d\mathbf{l}\, ds'\, ds \approx -4\pi^2 r_0^4. \tag{28}$$

Thus (22)-(25) imply

$$L(\alpha+i\beta) \approx L_0 - i\left(\frac{4\mu_0 r_0^3}{3c^2}\alpha\beta+\frac{\pi\mu_0 r_0^4}{6c^3}\beta^3\right). \tag{29}$$

Using (6) to solve for $(\alpha+i\beta)^2$, (29) and (24) yield

$$\alpha^2-\beta^2\approx-\frac{1}{CL_0}=-\omega_0^2 \tag{30}$$

$$2\alpha\beta\approx-\frac{\omega_0^2}{L_0}\left(\frac{4\mu_0 r_0^3}{3c^2}\alpha\beta+\frac{\pi\mu_0 r_0^4}{6c^3}\beta^3\right). \tag{31}$$

(30) and (31) together imply that α satisfies

$$\alpha^2+\frac{6c^3}{\pi r_0^4\mu_0}\left(\frac{2L_0}{\omega_0^2}+\frac{4\mu_0 r_0^3}{3c^2}\right)\alpha+\omega_0^2\approx 0. \tag{32}$$

By (26) and the definition of ε, the second term in the parentheses is negligible compared to the first, so one can rewrite (32) as

$$\alpha^2+\left(\frac{12M^3L_0}{\varepsilon^3\pi r_0\mu_0}\omega_0\right)\alpha+\omega_0^2\approx 0. \tag{33}$$

Noting that the coefficient of α is $\gg 1$, one can use the quadratic formula (taking the term with the plus sign since $|\alpha|<M\omega_0$) and the approximation $\sqrt{b^2-x}\approx b-(x/2b)$ to obtain

$$\alpha\approx-\frac{\varepsilon^3\pi r_0\mu_0}{12M^3L_0}\omega_0. \tag{34}$$

Thus (15), (26) and (34) imply $|\alpha|\ll\omega_0$ and it follows from (30) that

$$\beta\approx\omega_0. \tag{35}$$

Using (34) and (35) in (18) one obtains the approximate solution current

$$I(t)\approx-Q_0\omega_0\,e^{\alpha t}\sin\omega_0 t. \tag{36}$$

Since $|\alpha|\ll\omega_0$ it is interesting to view (18) and (36) from the perspective of an underdamped $LCR_{\rm rad}$ circuit in which $C=$ capacitance of the parallel-plate capacitor, $L=C^{-1}(\alpha^2+\beta^2)^{-1}\approx L_0$ and

$$R_{\rm rad}=2\alpha L\approx 2\alpha L_0. \tag{37}$$

Using (34) in the expression for $R_{\rm rad}$, one has

$$R_{\rm rad}\approx\frac{\pi\mu_0 r_0^4\omega_0^4}{6c^3}. \tag{38}$$

The right-hand side of (38) is the radiation resistance of a small loop antenna driven at an angular frequency ω_0, see Becker and Sauter (1982). Thus, one can approximate the solution current with the current of an $L_0CR_{\rm rad}$ circuit in which L_0 is the geometric self-inductance of the wire loop and $R_{\rm rad}$ is the radiation reisstance of the loop driven at $\omega_0=(L_0C)^{-1/2}$. The approximation improves with smaller values of ε.

For a numerical example, if $C=50\mu$F, $r_0=0.05$m and $r_1=0.0005$m, then one has $L_0\approx3.10\times10^{-7}$ H and $\omega_0\approx2.54\times10^5\,\mathrm{s}^{-1}$. It is easy to check that r_0 satisfies condition (16) and that $\varepsilon\approx8.3\times10^{-5}\ll1$. The solution current is

$$I(t)\approx-2.5\times10^5Q_0\,e^{-1.0\times10^{-9}t}\sin 2.5\times10^5t. \tag{39}$$

From (38) the radiation resistance is

$$R_{\rm rad}\approx6.3\times10^{-16}\ \text{ohm}. \tag{40}$$

The current in (39) would decay to half of its maximum amplitude after about 22 years.

4. Concluding remarks

This work can be easily generalized to include the case of a normally conducting wire. The term '$-\oint \boldsymbol{E}\cdot d\boldsymbol{l}$' replaces '0' on the left-hand side of (1). Using Ohm's law and arguing as one did for (3), one derives

$$0=\frac{\mu_0}{4\pi(\pi r_1^2)^2}\int_S\int_S\oint\oint\frac{\ddot{Q}(t-rc^{-1})}{r}\,d\boldsymbol{l}'\cdot d\boldsymbol{l}\,ds'\,ds+R\dot{Q}(t)+C^{-1}Q(t). \qquad (3')$$

Here R denotes the electrical resistance of the wire loop. To solve (3′) one looks for functions $Q(t)=e^{zt}$ where $z=\alpha+i\beta$ satisfies

$$L(z)z^2+Rz+C^{-1}=0. \qquad (6')$$

Comparing the functions

$$f(z):=L_0z^2+Rz+C^{-1}$$
$$g(z):=L(z)z^2+Rz+C^{-1} \qquad (8')$$

on a suitably large circle centred at the origin, Rouché's theorem will imply that g has exactly two zeros (counting multiplicity) enclosed within the circle if r_0 is not too large. However, the analysis is more involved than before as one must consider the various possibilities for the zeros of the polynomial $f(z)$. By way of example, the case of complex conjugate roots is outlined here.

Note that the complex roots of f have modulus ω_0. It is easy to see that any zeros of $g(z)$ must be complex and that they occur in conjugate pairs. Reasoning as before, one concludes that (9) and (10) are satisfied if $N>M>1$ and

$$\varepsilon<\frac{M}{2N}\ln\left(\frac{N-M}{M^3}\left(M^2-\frac{R}{L_0\omega_0}M-1\right)\right). \qquad (14')$$

(It is understood that N and M are chosen such that the right-hand side of (14′) is positive.) Thus, there are two distinct solutions, $Q_1(t)=e^{zt}$, $Q_2(t)=e^{\bar{z}t}$, $|z|<M\omega_0$ of (3′) if

$$r_0<\frac{c}{2\omega_0N}\ln\left(\frac{N-M}{M^3}\left(M^2-\frac{R}{L_0\omega_0}M-1\right)\right). \qquad (16')$$

If ε is small enough, say less than $\pi/4$, then it follows from (5) and (6′) that $\alpha<0$, i.e. the current decays as it oscillates. The uniqueness of the solution of the initial value problem can be deduced by employing the Laplace transform.

The approximation argument is very much the same as before, except that to determine α and β one must place (29) in (6′) instead of (6). Rather than repeating this argument, it is interesting to see how the effects of radiation resistance compare to those of electrical resistance, so a different approximation is outlined.

Let $-\delta$ denote the imaginary part of (29) and assume that $\varepsilon\ll 1$. Thus, δ is of the order $\mu_0r_0\varepsilon^2\ll L_0$. From (6′) one has

$$z\approx\frac{-R+i\sqrt{(4C^{-1}L_0-R^2)-i4C^{-1}\delta}}{2(L_0-i\delta)}. \qquad (I)$$

Taking the root of the complex number and using $\sqrt{b^2+x}\approx b+x/2b$, one obtains

$$\sqrt{(4C^{-1}L_0-R^2)-i4C^{-1}\delta}\approx\sqrt{(4C^{-1}L_0-R^2)}-i\frac{2C^{-1}\delta}{\sqrt{4C^{-1}L_0-R^2}}. \qquad (II)$$

Let ω^0 denote the quantity $(\omega_0^2-(R/2L_0)^2)^{1/2}$. Then from (I) and (II),

$$\alpha \approx -\frac{1}{2L_0}\left(R+2\delta\omega^0-\delta\frac{\omega_0^2}{\omega^0}\right)$$

$$\beta \approx \omega^0-\frac{R}{2L_0^2}\delta+\frac{\omega_0^2}{2L_0^2\omega^0}\delta^2. \qquad \text{(III)}$$

Referring back to the numerical example, a copper wire with the stated dimensions has a resistance $R\approx 8\times 10^{-3}$ ohm (Powell 1979). Condition (16′) is satisfied with $N=3M=6$, and since $\varepsilon\approx 8.3\times 10^{-5}$ and $\omega^0\approx\omega_0$, one sees that

$$\alpha \approx -\frac{R}{2L_0}\approx -1.3\times 10^4\ \mathrm{s}^{-1}$$

$$\beta \approx \omega^0 \approx 2.5\times 10^5\ \mathrm{s}^{-1}. \qquad \text{(IV)}$$

These are the values one would expect from an L_0RC circuit; the effects of the radiation resistance are negligible compared to those of the electrical resistance.

The work in this paper can also be generalized to approximate the case of a superconducting loop of wire. The idea is to account for the surface current by regarding the wire as a hollow, perfectly conducting tube and adjusting integrations accordingly. Instead of (3), one derives

$$0=\frac{\mu_0}{4\pi(2\pi r_1)^2}\int_T\int_T\oint\oint\frac{\ddot{Q}(t-rc^{-1})}{r}\,\mathrm{d}\boldsymbol{l}'\cdot\mathrm{d}\boldsymbol{l}\,\mathrm{d}s'\,\mathrm{d}s+C^{-1}Q(t). \qquad (3'')$$

Here T denotes the circle of radius r_1 which is the cross-section of the hollow tube. One difference in this setting, for example, is the lower geometric self-inductance of the loop:

$$L_0\approx\mu_0 r_0\left[\ln\left(\frac{8r_0}{r_1}\right)-2\right] \qquad r_1\ll r_0. \qquad (26'')$$

As a final remark, note that this work can also be applied to the popular two-capacitor problem (Powell 1979), providing a rigorous demonstration of the behaviour of the two-capacitor circuit in the absence of electrical resistance.

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